7^{th} CCSS D1 Ratios and Proportional Relationships Conceptual Foundation (approximately 8 weeks)

Domain 1: Ratios and Proportional Relationships 7.RP

D1 Cluster1: Analyze proportional relationships and use them to solve real-world and mathematical problems.

1. Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units
2. Recognize and represent proportional relationships between quantities
   a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.
   b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
   c. Represent proportional relationships by equations. For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as t = pn
   d. Explain what a point (x,y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0,0) and (1,r) where r is the unit rate.
3. Use proportional relationships to solve multistep ratio and percent problems.

1.I.1 1 can draw a model for a proportional relationship and connect it to an equation to solve a proportion.
1.1.2 I can compute unit rates with ratios of fractions, including lengths, areas and different units.
1.1.3 I can determine whether two quantities are proportional from either a table or graph.
1.1.4 I can identify the unit rate in tables, graphs, equations, diagrams, and verbal descriptions.
1.1.5 I can represent proportional relationships by equations.
1.1.6 I can interpret and explain what a point (x,y) means on a proportional graph, paying special attention to (0,0) and (1,r), where r is the unit rate.
1.1.7 I can use proportions to solve multi-step ratio and percent problems, including interest, tax, discounts, tips.
*3.1.5 I can restate expressions to make sense of real-life situations. (for example, the perimeter of a rectangle can be l+l+w+w or 2l+2w)

*Please note: The following conceptual foundation highlights important concepts and instructional support, but is not all inclusive. Please refer to “I Can” statements above for a more complete picture of content.

What is a ratio? A ratio is a relationship between two quantities. A ratio can be written as A:B, A/B or by the phrase "A to B". **It is important to understand that a ratio can be a "part to whole" OR "part to part" a relationship**. (See the 6th Grade Math Ratio Conceptual Foundations document for additional information on ratio). EXAMPLES:

Part to whole relationship

There are men and woman in a meeting. If the ratio of woman to people in a room is 3 to 5 and there are 40 people in the room, how many women are in the room?

ANSWER: 24 woman

Part to whole: woman to people
3:5 is 3 woman: 5 people OR
3/5 is 3 woman/5 people (this CAN be thought of as a fraction because it is a part to whole relationship)

TABLE REPRESENTATION

Woman	People	Men
3	5	2
6	10	4
9	15	6
12	20	8
15	25	10
18	30	12
21	35	14
24	40	16
27	45	18
30	50	20

From the table we see that when there are 24 women, there are 40 people. We also can see that if there are 24 woman, there are 16 men, hence a total of 40 people. RATE can be discussed from the table in a number of ways: 5/3 people per woman; 3/2 woman per men; 5/2 people per men…all of which are related to the original part to whole ratio (but it informs a number of other part to part or part to whole ratios. NOTE that for "rate" we always say "output" per "input".

Part to part relationship

There are men and woman in a meeting. If the ratio of woman to men is 3 to 5 and there are 40 people in the meeting, how many women are in the room?

ANSWER: 15 woman

Part to part: men to woman
3:5 is 3 woman to 5 men OR
3/5 is 3 woman to 5 men (this CANNOT be thought of as a fraction because it is a part to part relationship)

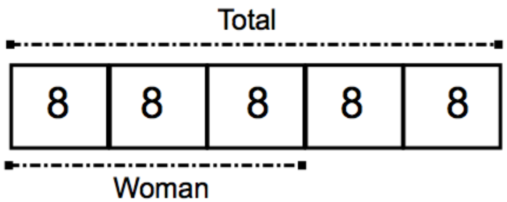
TABLE REPRESENTION

Woman	Men	People
3	5	8
6	10	16
9	15	24
12	20	32
15	25	40
18	30	48
21	35	54
24	40	62
27	45	70
30	50	78

From the table we see that when there are 40 people, there are 15 woman and 25 men. Notice that in this example, although we used the "same" ratio a) there are always more men than woman in this case, and b) the answer is different than in the last example. The reason the answer is different is because although 3 still represented woman, the 5 is the number of men NOT the total of people. As with the previous example, there are now a variety of rates (ratios) we can discuss.

PICTORIAL REPRESENTATION

There are 40 people, and the ratio of woman to people is 3/5. Thus the whole is divided in to 5 parts and 3 of them are woman. 40/5 is 8, hence there are 8 in each “section”.



Thus for 40 people and a ratio of 3 woman per 5 people, there are 24 woman in the room.

Notice that 3/5 of 40 is 24. We can think of the ratio in this case as a fraction because it is a *part to whole* relationship.

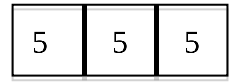
How is the model related to an algebraic equation?

From the model, a student should see that woman (w) are 3/5 of the total (40). Thus woman w=(3/5)(40). Additionally, students should recognize that in finding the number of woman we first divided the total into 5 equal parts and then 3 of the parts was the woman.

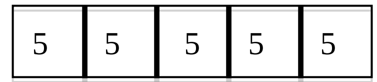
PICTORIAL REPRESENTATION

Now the 40 people are evenly distributed among 3 units of woman and 5 units of men. Thus the total number of units is 8 and if we evenly divide 40 into 8 units, each unit has 5.

Woman



Men



Thus if the ratio of woman to men is 3 to 5 and there are 40 people, there will be 15 woman.

Notice that in this case 3/5 is not a fraction as before. However because this was a *part to part* relationship. Also note that the 3/5 told us that the total of units in this case was 8.

How is the model related to an algebraic equation?

From the model, students should see that there are 3 parts woman and 5 parts men. Because each of the parts are the same, we can say woman are “3x” and men are “5x”. The total (3x+5x) is 40. Thus 3x+5x=40, x=5. But each of the x values is one part. We need 3 for woman (3x5) and 5 for men (5x5).

What is a proportion?

A proportion is an equality between two ratios. By this, we mean that the ratio on both sides of the equation represent the same quantity types in the same manner. For example: 3 cat to 4 dogs =12 cats to 16 dogs.

Proportions are used to find missing parts of ratios. Students often have difficulty setting up proportions because they do not carefully consider the meaning of the quantities with which they are working and therefore often set up false equalities.

EXAMPLE PROBLEM: There are 40 people in a room. If the ratio of woman to people in the room is 3/5, how many men and woman are there in the room?

In this situation, a student many set up a correct proportion: $\frac{3}{5}=\frac{x}{40}$, and find that x=24, but they may not understand what the "24" means, or how to use the 24 to find the number of men.

A more difficult but related problem: There are 40 people in a room. If the ratio of women to men is 3/5, how many women and men are in the room?

Some students will again want to use the proportion: $\frac{3}{5}=\frac{x}{40}$, but in this situation, the proportion is **NOT** correct. The left is showing the relationship between women and men, but the right is (presumably) woman to total people.

$\frac{woman}{total}=\frac{3}{3+5}$. Therefore: $\frac{3}{3+5}=\frac{x}{40}$. The solution to proportion is x=15. Now students have to understand that x is the number of woman, if the total is 40. Hence 40-15=number of men=25.

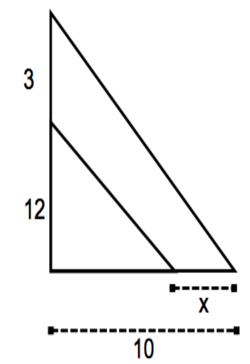
Each of these problems can also be solved (as shown above) using models. We often teach students to "cross-multiply" to solve a proportion. Although this method "works" it is a) important that students understand why it "works" and b) understand why they cannot apply "cross-multiplication" to solve $\frac{3}{5}+x=\frac{2}{x}$. Thus it is better to say, "clear fractions" rather than to teach "cross multiply."

GEOMETRY Example Problem:

Setting up a proportion to find the value of x for picture on the right can be difficult for students. It is important that teachers start by a) reminding students that a proportion is equal ratios and b) there are many equal ratios that can be written for this situation.

A common mistake here would be to write: $\frac{3}{12}=\frac{x}{10}$ (or some similar incorrect relationship.) A good way to help student recognize the error is to have them use words like "little-part," "big-part," or "whole". Hence in the incorrect proportion above we have: $\frac{little-part}{big-part}=\frac{little-part}{whole}$, the ratios are not equal.

In this problem, the value of x can be found directly (x=2). In more difficult problems, students may find the value of x and then use it to find other values (the side corresponding to 12 is equal to 10-2 or 8).



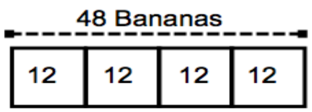
Using Proportion to Solve Percent Problems

Percent problems are specific kinds of proportion problems. Students began their work with percent problems in 6th Grade Math. 7th Grade Math expends their understanding and fluidity with percent problems. Students should be reminded that: a percent is a "part whole" relationship wherein the whole is always 100.

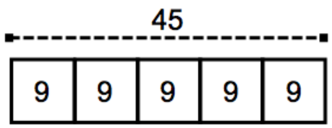
Because students will be confronted with percent type problems in every aspect of their lives, it is important that students gain fluency with percent problems without calculators. To do this a) students have to gain a conceptual understanding of percent and b) learning is scaffolded deliberately.

Benchmark Percent Amounts :Benchmark percent amounts should include: 10%, 25%, 50%, and 20%

In 6th Grade Math, students work with models to find benchmark percent amount. For example: 25% of Mary's 48 bananas have spoiled. How many bananas have spoiled?



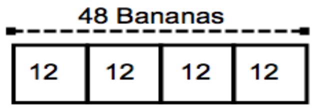
Students should start by thinking; "what portion of 100 is 25?" They should readily know that there are 4 groups of 25 in 100. Hence 25% of 48 is ¼ of 48. The bar model to the right models 48 divided into four equal parts.



This same strategy can be used to find 20% of 45. Students start by thinking about what part of 100 is 20, and then realizing that there are 5 groups of 20 in 100. Hence, 45 must be divided into 5 equal parts with 9 in each.

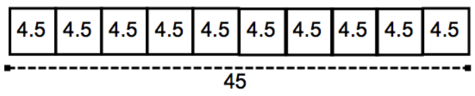
"Parts" or "Sums" of Benchmark Percent Amounts: Students move to parts and/or sums (half, double, triple etc.) of benchmarks next. These generally include 5%, 1%, 40%, 60%, 80%, 75%

75% of Mary's 48 bananas have spoiled. How many bananas have spoiled? Students should start by thinking about how they can "get" to 75 out of 100. This should lead them to realize that 75% is 3 groups of 25%. With the model, they can see that 75% of 48 is 36.



Now let's extend this same kind of thinking to values that may or may not be whole numbers.

John's dog, Bob, weighs 45 pounds, the doctor insists that it lose 10% of its weight. How much weight does the doctor want Bob to lose? In the model, we divided 45 into 10 equal parts and can see that one part is 4.5 pounds, hence, the dog needs to loose 4.5 pounds.

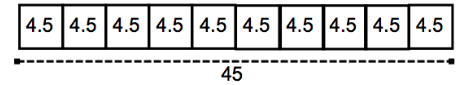


John put the Bob on a strict diet, but Bob has been secretly eating the cat's food. When they return to the doctor, John learns that Bob has only lost 5% of his weight. How much weight did Bob loose? Here, students should understand that 5% is half of 10%, hence we are looking for half of 4.5 or 2.25.

Benchmark +"Part" Amounts: Students should move to amounts that require them to put amounts together

Find 35% of 45:

Using the model we used for the "Bob" problem above, students should reason that 35% of 45 is 3 groups of 10% (4.5) and ½ a group of 10% (2.25). Hence, 35% is 4.5+4.5+4.5+2.25



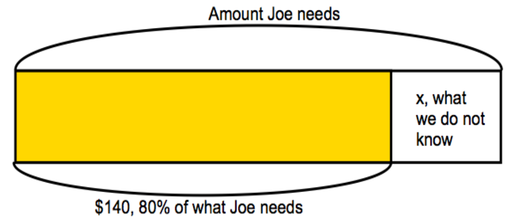
Multi-step Percent Problems

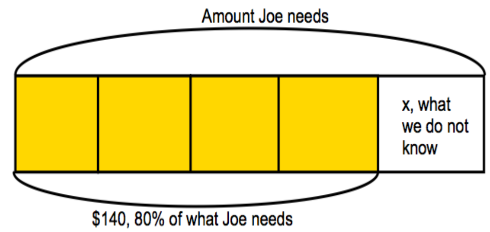
Models are excellent tools to help students connect concrete to algebraic representations.

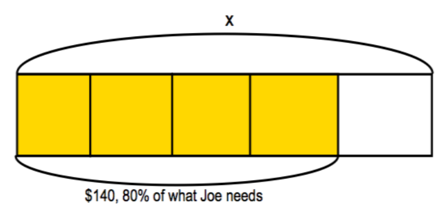
Example:
Joe 80% of the money he needs to by the new phone he wants. If he has $140 in his savings account, how much more money does he need?

A student might draw the picture to the right. The picture shows that 20% of the money Joe needs is missing. So one way the student might think of solving the problem would be to divide $140 into 4 equal parts, and then multiply one part by 5 to get the total (140/4=35, hence the missing piece is $35 or the whole is $175.)

Using the same picture, the student might see that $140 is 80% of x, thus: 140=.80x



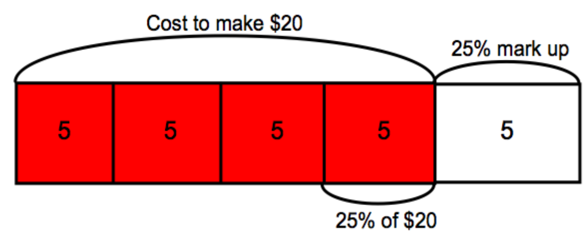




Example:

Jesus makes phone rubber protective cases. Each case costs $20 to make. If he wants to make a 25% profit, how much should charge for phone cases?

This model helps students see that first we need to find 25% of $20 and then add it to $20 or .25(20)+20=cost i.e. 1.25(20)=cost.



UNIT RATES

The relationship between ratio, proportion, slope and unit rate is often not explicitly addressed with students. It is important that teachers help student make connections among these concepts.

Example:
In 4 hours Lisa can mow 7 lawns.

a) What is the ratio of hours to lawns mowed?
b) How many lawns can Lisa mow in 8 hours and in 10 hours?
c) How many lawns can she mow in 1 hour?
d) Write an equation for how many lawns she can mow in any number of hours.

Solution:
It is vitally important that throughout this type of exercise students *attend to precision* in language i.e. students attend carefully to the meaning of the questions and the related values.

A student may start by creating a table of values:

Hours	Lawns
0	0
1	
2	
3	
4	7
5	
6	
7	
8	14
9	
10	
11	
12	21

The table to the right has hours as the "input" value and "lawns" as the "output" value, students may or may not set up their table this way (i.e. they may have lawns as the input and hours as the out put.) With hours as the input, for hour multiples of 4, lawns are multiples of 7. For hour values that are not multiples of 4, lawn values may seem difficult for students to compute.

Question "a" is evident without the table: The question is asking for the ratio of hours to lawns, thus 4 hours to 7 lawns or 4/7 hours per lawn.
Note here that the ratio is set up as hours to lawns. This is also how the table above is set up. Had the student set the table up the other way, he or she can still easily deduce the ratio from the table, but would have to reverse the order for the answer from the way it appears in the table.

Question "b": from the table, we can see that in 8 hours, Lisa can mow 14 lawns. The table does not have a value for 10 hours. Another way a student might solve the problem is to set up a proportion:

$$\frac{4 hour}{7 lawn} = \frac{8 hours}{x lawn} \quad \text{and} \quad \frac{4 hour}{7 lawn} = \frac{10 hours}{x lawn}$$

Using the proportions, for 8 hours we again get 14 lawns, and for 10 hours we get 70/4 or 35/2 or 17.5 lawns in 10 hours.

Question "c": again the table does not have a value, so again we can use a proportion to solve. $\frac{4hour}{7lawn} = \frac{1hours}{xlawn}$

In solving the proportion we get that in 1 hour 7/4 or 1.75 lawns can be mowed. Let's write that answer as the ratio was expressed above (hours/lawns)

$\frac{4hour}{7lawns} = \frac{1hour}{xlawns}$ thus $\frac{1hour}{\frac{7}{4}lawns}$. ***It is important to remember that rations are not (necessarily) fractions. In other words IF $\frac{1}{7/4}$ is a fraction than it is equal to $\frac{4}{7}$. But in this case, $\frac{1}{7/4}$ is the ratio of $\frac{hours}{lawns}$. Hence if we want the ratio of $\frac{lawns}{hours}$ than it is represented as $\frac{7/4}{1}$.

"Unit Rate" is a rate per one unit. So in this case $\frac{7/4}{1}$ or $7/4$ is the unit rate. In other words: $7/4$ lawns per 1 hour.

Question "d": The equation is specifically asking for "how many lawns she can mow in any number of hours" or lawns per hour. Hence "lawns" is the output and hours is the input or y/x. To write an equation we need a point and a slope. There are several points ((0,0), (4,7), (8,14)) identified by the table. The slope will be in terms of lawns per hour. To find the slope we need to find the change in the output over the change in the input. Well, we can use the points (4,7) and (8,14) and get $\frac{14 - 7}{8 - 4}$ or $\frac{7}{4}$ as the slope. (Notice that the rate, unit rate and ratio of lawns to hours are all the same. This is an important connection to help students make). The easiest point to use from above is (0,0) because it is the y intercept, thus the equation is y=$\frac{7}{4}$x.

As you can see from the table and equation, each time we increase by an hour, we increase 7/4 lawns. Hence the unit rate is the same as the slope. In other words, whether the input increases by 1, 4, 5, 7…any value, the ratio of lawns to hours is always 7/4. Another way students may see this is that we are "counting" by 7/4.

Hours	Lawns
0	**0**
1	**7/4**
2	2(7/4) or **14/4**
3	3(7/4) or **21/4**
4	4(7/4) or **28/4 or 7**
5	5(7/4) or 35/4
6	6(7/4) or 42/4
7	7(7/4) or 49/4
8	**8(7/4) or 56/4 or 14**
9	9(7/4) or 63/4
10	10(7/4) or 70/4
11	11(7/4) or 77/4
12	**12(7/4) or 84/4 or 21**

More Unit Rate

In the previous example, we associated unit rate with a table. Now let's look at different rates not presented as a unit rate and find the unit rate.

Mary can run 1 mile in 7 minutes, how far can she run in one minute and in one hour?

Solution:

miles/minute= $\frac{1}{7}=\frac{x}{1}$ Hence, x=1/7 or the unit rate is $\frac{1/7}{1}$

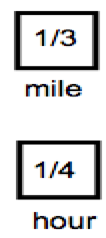
Iterating rates to find unit rates can seem difficult to students unless they have a strong conceptual understanding of rate:
Example:

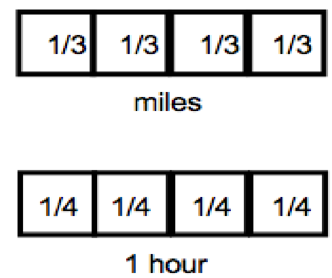
Rico has an old dog, Julio. His dog can only walk 1/3 of a mile in 1/4 of an hour. How many miles can he walk per hour?

In 6th grade, students learned to draw models of similar problems:

For every 1/3 mile Julio walks, 1/4 hours pass.

Because we need four ¼ hours to get a full hour, and in every ¼ hours Julio walks 1/3 miles, in one hour, Julio will walk 4.3 miles or 1 and 1/3 miles. Hence, Julio walks 4/3 miles





Solving this same problem as a proportion (mile/hours):

$$\frac{1/3}{1/4}=\frac{x}{1}$$

$$\frac{1}{4}x=\frac{1}{3}$$

$$x=\frac{4}{3}$$

Here it is important for students to remember what the x represents. The ratio was miles per hour, the x represented an unknown quantity of miles.

Another way for students to think about this is again to "clear fractions" i.e. multiply both the numerator and the denominator by 4.

Determining if two quantities are proportional from a table or a graph

Students have had some experience with tables and graphs in 6th Grade Math. In 7th Grade Math students learn to draw conclusions from tables and graphs.

Example:
For the following tables of values, determine if the quantities are proportional.

Day	Money Earned
1	3
2	6
3	9
4	12

In this example, we can see that each day the amount of money earned increases by 3. Hence the rate is 3 dollars per.

Example

Day	Money Earned
1	1
2	2
3	4
4	8

In this example, we can see that each day the amount of money earned doubles from the previous day

Proportion/Unit Rates/Equations

As students study patterns, they will identify the unit rate (slope). Throughout this learning series, they should continually push toward the equation which describes the growth or change. For example, in the equation D=RT, "the distance is proportional to the time spent" or "for every hour, you can travel a certain distance" or "Distance/Time = Rate "

Examples:

Story	Equation	Explanation
Chris can ride a bike 5 miles per hour	$y = 5x$	"Distance (*y*) is proportional to time spent (*x*)." "Unit rate is 5miles/1hr" –(for every hour, Chris can travel 5 miles). "The distance (*y*) divided by the time (*x*) gives you the proportion or unit rate"
Tom can ride a bike 9 miles in 2 hours	$y = 4.5x$	"Distance (*y*) is proportional to time spent (*x*)." "Unit rate is 4.5miles/1hr" –(for every hour, Tom can travel 4.5 miles). "The distance (*y*) divided by the time (*x*) gives you the proportion or unit rate"
Christie can ride her tricycle 1 mile in 4 hours	$y = .25x$	"Distance (*y*) is proportional to time spent (*x*)." "Unit rate is .25miles/1hr" –(for every hour, Tom can travel 1/4 mile). "The distance (*y*) divided by the time (*x*) gives you the proportion or unit rate"
Paul gets paid $10 per hour.	$y = 10x$	"Amount earned (*y*) is proportional to time spent (*x*)." "Unit rate is $10 /hr" –(for every hour, Paul earns $10.). "The amount earned (*y*) divided by the time (*x*) gives you the proportion or unit rate"
Val gets paid $5 expense money every day plus $12 every 2 hours.	$y = 6x + 5$	"Amount earned (*y*) is not directly proportional to the time spent (*x*)—because of the extra expense money." While Val does earn $6 per hour, his total earnings (*y*) is not directly proportional to the time spent.
A dog eats a 15 pound bag of dog food in 3 weeks.	$y = 3x$	"Amount of food (*y*) is proportional to weeks (*x*)." "Unit rate is 5 pounds/week" –(for every week, the dog eats 5 pounds of food). "The amount of food (*y*) divided by the weeks (*x*) gives you the proportion or unit rate"

Identifying proportion from data, graphs, equations

1. Examine the data samples in the tables below. Is the data proportional? If so, what is the unit rate?

	Proportionate (yes or no)	Unit rate?
Money Saved	yes	$5/week
Tearing Paper		
Area of a Square		
Traveling	yes	25miles/hr

Data Samples

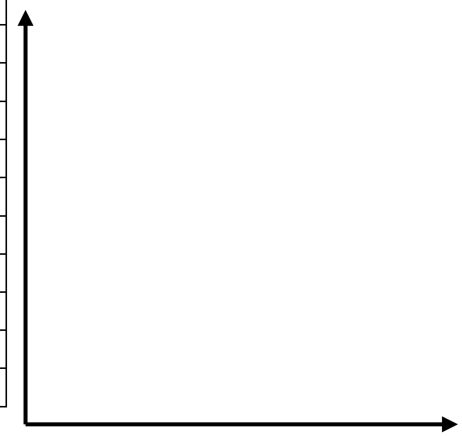
Money Saved			**Tearing Paper**			**Area of Squares**			**Traveling**	
Weeks	$ Saved		Tears	Pieces		Length	Area		Hours	Miles
1	$5		0	1		1	1		2	50
2	$10		1	2		2	4		4	100
3	$15		2	4		3	9		6	150
4	$20		3	8		4	16		12	300
n	5n		n	2^n		n	n^2		n	25n

2. In the data table above, fill in the empty cell—what is the symbolic expression to describe the pattern in each table?

3. Proportion found in growth patterns

Is the growth proportional, yes or no?
What is the unit rate
What equation describes the perimeter at any stage?
Describe how you see the unit rate (proportion) on the graph.

Stage # or area	Perimeter
1	3 units
2	4 units
3	5 units
4	6 units
5	7 units
10	12 units
100	102 units
x	x+2 units



4. Find the proportion (unit rate) in the pattern. Write the equation.

1) x	y	2) x	y	3) x	y
1	2	2	6	2	6
2	4	3	9	4	10
3	6	4	12	6	14
10	____	10	____	10	____

1)	2)	3)
What is the unit rate?	What is the unit rate?	What is the unit rate?
y = ______________	y = ______________	y = ______________

5. Proportion or unit rate from a graph.
    a. Which graphs show proportional relationships?
    b. What is the unit rate in those graphs? Explain how you know what the unit rate is.